

A MATHEMATICIAN AT PLAY

# Even numbers can be unfair

Have you been introduced to modular arithmetic? A system of arithmetic for integers wherein numbers wrap around upon reaching a certain value, which is called the modulus, number circles are the norm in modular arithmetic. **Daniel Finkel** introduces you to this concept by defining a game where numbers can be fair or unfair...

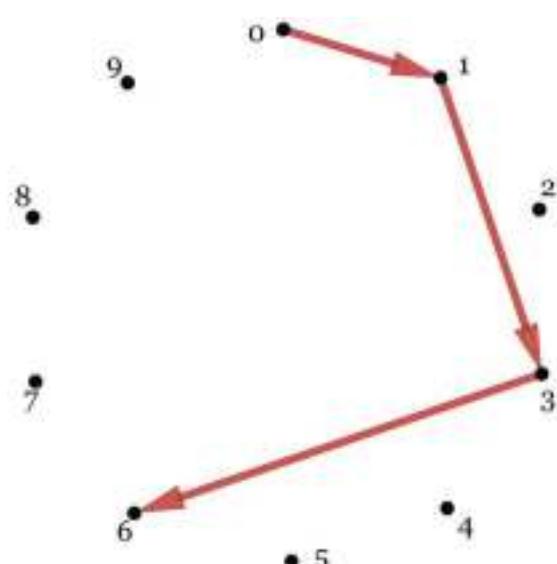
Here's a lovely problem with a structure that surprises. Imagine a group of people in a circle, passing a ball around clockwise. The first person passes the ball to the person next to them, which I'll call 'forward one space.' That person passes the ball forward two spaces, and that person passes the ball forward three spaces, and so on.

Does the ball reach everyone in the circle?

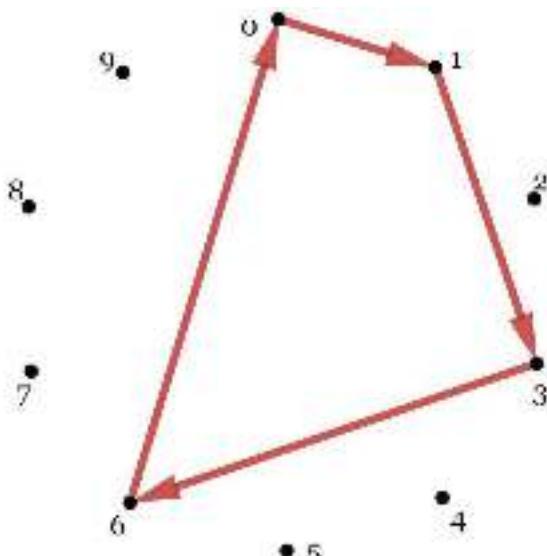
If it does, let's call the number of people in the circle **fair**, and if not, we'll call the number **unfair**.

At first glance, it seems like the ball must reach everyone, and that every number should be fair. After all, the numbers go on forever. What would stop it from getting everywhere? But let's look at an example and see why it might fail.

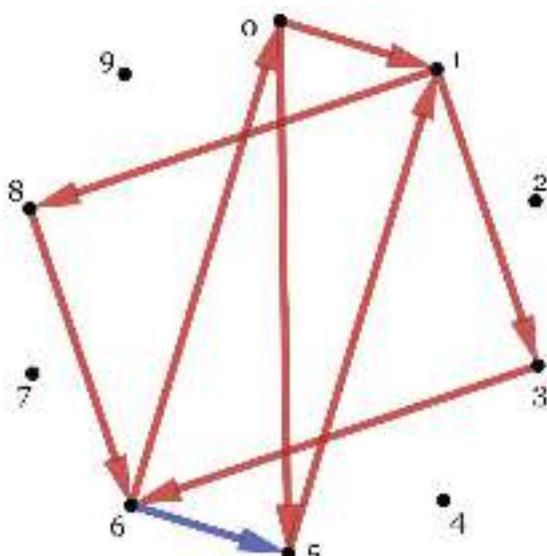
Imagine ten people in a circle. The first person throws the ball forward one space. Then it travels forward two spaces, and three (shown below).



So far so good. But now, when it goes forward four spaces, it will return to where it started.



Will it be trapped in some loop and forever miss people as it travels in the circle? At first, it doesn't seem so. It goes forward, hitting new spots and repeating old ones in the circle.



At this point, having travelled nine spots forward, or one back, in the path coloured blue, the next throw is ten forward, which is the same as being thrown in place! Then eleven forward, which is the same as travelling one space forward. Suddenly, we're travelling the same path we took before, except in reverse! From then on, the ball is stuck on that path. It lands on just the same six spots, and misses the other four forever.

In other words, **10 is an unfair number**.

So that's the big question: what numbers of people can we put in the circle so that everyone will touch the ball? In other words, what numbers are fair, and what numbers are unfair?

**PUZZLE 1**

There is just one fair number between 10 and 20. What is it?

**PUZZLE 2**

There is just one fair number between 20 and 50. What is it?

**PUZZLE 3**

Can you find a fair number greater than 50?

How can you tell if a number is fair or not without the tedious work of drawing out the entire problem?

Dan Finkel is the founder of Math for Love, an organisation devoted to transforming how math is taught and learned. He is the creator of mathematical puzzles, curriculum, and games, including the best-selling *Prime Climb* and *Tiny Polka Dot*.